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THE SECOND HYPERBOLIC INTEGRAL.

By F. P. MATZ, M. Sc., Ph. D., New Windsor, Maryland.

From $y^2=(e^2-1)(x^2-a^2)$, $dy/dx=x\surd[(e^2-1)/(x^2-a^2)]$. Hence the hyperbolic arc bounding a segment symmetrical with respect to the major axis of the hyperbola, becomes

$$s=2\int_0^{x'}\sqrt{\left(1+\frac{(e^2-1)x^2}{x^2-a^2}\right)}dx=2\int_0^{x'}\sqrt{\left(\frac{e^2x^2-a^2}{x^2-a^2}\right)}dx\ldots.(1).$$

Transforming (1) by means of the *circular* notation, $x=a\sec\theta$ and $y=b\tan\theta$, we obtain

$$s=2a\int_0^{\theta'}\sec\theta\surd(e^2\sec^2\theta-1)d\theta,\ =2ae\int_0^{\theta'}\sec^2\theta\sqrt{\left[1-\left(\frac{1}{e^2}\right)\cos^2\theta\right]}d\theta\ldots.(\alpha).$$

Let $c^2=1/e^2$; that is, $c<1$ and $e>1$. The amplitude of an hyperbolic integral may be $\frac{1}{2}\pi$; but the instant the amplitude exceeds $\sec^{-1}(e)$, the integral transcends the confines of the real. From (α), therefore, we have

$$s=2ae\int_0^{\theta'}\left[\sec^2\theta-\frac{c^2}{2}\left(1+\frac{c^2}{4}\cos^2\theta+\frac{c^4}{8}\cos^4\theta+\text{etc.}\right)\right]d\theta$$

$$=2ae\left[\tan\theta-\frac{c^2}{2}\int\left(1+\frac{c^2}{4}\cos^2\theta+\frac{c^4}{8}\cos^4\theta+\text{etc.}\right)d\theta\right]_0^{\theta'},=\mathbf{H}''(c,\theta')\ldots.(2),$$

which is an hyperbolic integral of the second order and expressed by the circular notation.

[Since $dx=a\sec\theta\tan\theta d\theta$ and $dy=b\sec^2\theta d\theta$, the expressions for s in (α) are easily deduced by a *second* method.]

Integrating (2) between the limits of $\theta=0$ and $\theta=\sec^{-1}(e)$,

$$s=2ae\left\{e\sqrt{\left(1-\frac{1}{e^2}\right)}-\frac{1}{2e^2}\left[1+\frac{1}{8e}\left(1+\frac{3}{8e^4}\right)\sqrt{\left(1-\frac{1}{e^2}\right)}\right.\right.$$

$$\left.\left.+\frac{1}{8e^2}\left(1+\frac{3}{8e^2}\right)\sin^{-1}\left(\sqrt{1-\frac{1}{e^2}}\right)\right]\right\},=\mathbf{H}''[c,\sec^{-1}(e)]\ldots.(3),$$

a symmetrical formula giving a very approximate numerical result. Put $a=1$, and $e=2$; then $s=6.25648+$. According to the Lambertian system of notation for *Hyperbolic Functions*, since $x=a\sec\theta=a\cos U$ and $y=a\tan\theta=a\sin U$, we have from (α) the hyperbolic integral,

$$s=2a\int_0^{U'}\cos U\surd(e^2\cos^2U-1)\times\frac{\sin U dU}{\cos U\surd(\cos^2U-1)}$$

$$=2a\int_0^{U'}\sqrt{(e^2\cos^2 U-1)}\,dU=2a\sqrt{(e^2-1)}\int_0^{U'}\sqrt{\left[1+\left(\frac{e^2}{e^2-1}\right)\sin^2 U\right]}\,dU\ldots(\beta).$$

Make $C^2=e^2/(e^2-1)$; then, after obvious reductions,

$$s=2a\sqrt{(e^2-1)}\int_0^{U'}\sqrt{(1+C^2\sin^2 U)}\,dU=2a\sqrt{(e^2-1)}\int_0^{U'}\left[1+\frac{C^2}{2}\sin^2 U\right.$$

$$\left.-\frac{C^4}{8}\sin^4 U+\frac{C^6}{16}\sin^6 U-\text{etc.}\right]dU,\ =\mathbf{H}''(C,\ U');\text{or}\,\mathbf{H}''[C,\cosh^{-1}(e)]\ldots(4),$$

which are forms (1) for the *incomplete* and (2) for the *complete-real* hyperbolic-integral of the second order expressed by the hyperbolic notation of Lambert.

[Since $dx=a\sin U dU$ and $dy=b\cos U dU$, the expressions for s in (β) are easily deduced by a *second* method.]

The analogy between θ and U is expressed through the areas of the circular and hyperbolic sectors. This makes θ the hyperbolic amplitude of U; and this amplitude our former teacher, Professor Cayley, tersely denominated the *Gudermannian of U*.

Remembering that Exp. $U=\cos U+\sin U=\sec\theta+\tan\theta$, we have $U=Gd^{-1}(\theta)=\log(\sec\theta+\tan\theta)=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)\ldots(\nu)$.

From trigonometrical tables, by means of (ν), the value of U as a function of θ can easily be calculated. Obvious operations, also, give

$$s=2\int_0^{y'}\sqrt{\left(1+\frac{y^2}{(e^2-1)(b^2+y^2)}\right)}\,dy=\frac{2}{\sqrt{(e^2-1)}}\int_0^{y'}\sqrt{\left(\frac{e^2y^2+a^2(e^2-1)^2}{y^2+a^2(e^2-1)}\right)}\,dy$$

$$=2a\int_0^{\theta'}\sec\theta\sqrt{(e^2\sec^2\theta-1)}\,d\theta,\ =2a\sqrt{(e^2-1)}\int_0^{U'}\sqrt{\left[1+\left(\frac{e^2}{e^2-1}\right)\sin^2 U\right]}\,dU,$$

which, of course, is just as it should be.

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THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

CHAPTER II.

To determine the equation upon the solution of which depends the inscription of the regular polygon of n sides.

Let $na=\pi$. Here n is supposed to be odd. Write p for $\frac{n-1}{2}$. Then $\sin pa=\sin(p+1)a\ldots(3)$.